Using Graphs in Modeling the Movements of a Manipulation Robot

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Abstract. The manipulation robot consists of absolute rigid bodies connected to each other by cylindrical hinges, as well as to an external body performing a given movement. When describing the relationships between absolutely rigid bodies of a manipulation robot, a graph with a tree structure is used. Control actions are modeled by the moments of forces applied in cylindrical hinges. A technique is proposed that makes it possible to use packages of analytical calculations when compiling Lagrange equations of the second kind with a large number of degrees of freedom. The relevance of the technique is associated with its possible use in the construction of mathematical models of a wide class of manipulation robots used in technology, realizing the spatial movements of their grippers. These include industrial manipulators PUMA.

INTRODUCTION

Automation of the procedure for deriving dynamic equations for manipulation robots is an urgent task. It is useful to use graph theory [1, 2] when solving this problem. Fundamental results in this direction were obtained in the work [1]. In it, a complete solution of this problem is obtained for plane motions of mechanical systems with cylindrical joints. When deriving dynamic equations in this work, general theorems of the dynamics of mechanical systems were used. When applying the methods of analytical mechanics, technical difficulties arise associated with the mathematical description of the connections in a mechanical system. The indicated difficulties did not allow obtaining a complete solution of the indicated problem using the methods of analytical mechanics [2]. In this paper, we consider the spatial motions of a mechanical system with cylindrical joints. Holonomic constraints make it possible to use the Lagrange equation of the second kind when describing the dynamics of a mechanical system. For the case under consideration, in the present work, a complete solution of the above problem is obtained.

USING GRAPHS TO DESCRIBE LINKS

A mechanical system is considered, consisting of n absolute rigid bodies connected to each other by cylindrical hinges, as well as to an external body performing a given motion. The bodies of the mechanical system are associated with the vertices of the graph s_i, $i = 0, 1, ..., n$, and its hinges are the arcs of the graph u_a, $a = 1, ..., n$. The outer body corresponds to the vertex of the graph s_0. Mechanical systems for which the corresponding graphs have a tree structure are considered. We use the correct numbering of the vertices and arcs of the graph. The highest indices have the vertices farthest from the s_0 graph vertex. We choose the directions of the arcs to the s_0 vertex. When describing the relationships between bodies, we use the logistic functions i^+, i^-. The value of the function $i^+(a)$ is equal to the index of the vertex from which the arc u_a exits, and the value of the function $i^-(a)$ is equal to the index of the vertex into which the arc u_a enters. In a graph with correct numbering, the indices of the arcs u_a increase monotonically along the path from the vertex s_0 to the vertex S_i, $1 \le i \le n$. The numbering can be chosen in such a way that the equalities $i^+(a) = a$ and the inequalities $i^-(a) < a$ hold. For a simple open kinematic chain $i^-(a) = a - 1$, $a = \overline{1, n}$. The arc u_a corresponds to a cylindrical hinge that connects the bodies corresponding to the vertices of the graph with indices $i^+(a)$ and $i^-(a)$. When describing the relationships between bodies, the incidence matrix $S = \{S_{ia}\}_{i=\overline{1,n}}^{a=\overline{1,n}}$ is used, whose elements are defined by the formulas [1].

$$S_{ia} = \begin{cases} +1 & \text{if } i = i^+(a) \\ -1 & \text{if } i = i^-(a) \\ 0 & \text{in other cases} \end{cases}$$

For the graph under consideration, the matrix is upper triangular and all its elements on the main diagonal are equal to +1. The elements of the inverse matrix $S^{-1} = \{S^{-1}_{ia}\}_{i=\overline{1,n}}^{a=\overline{1,n}}$ are determined by the formulas:

$$S^{-1}_{ia} = \begin{cases} +1, & \text{if arc } u_a \text{belongs to the path between the vertices} s_0 \;\; s_i \\ 0, & \text{if arc } u_a \text{ not belongs to the path between the vertices} s_0 \;\; s_i \end{cases}$$

DETERMINING THE SPEEDS OF A MANIPULATING ROBOT

The movements of the manipulation robot are determined by relative rotations around the axes of the cylindrical joints. In this work, the directions of the cylindrical joints are determined by the unit vectors $\mathbf{e}_a$, $a = \overline{1,n}$, the directions of which can be arbitrary in inertial space. During motion, the position of the vector $\mathbf{e}_a$ does not change in the moving coordinate systems rigidly connected with adjacent bodies connected by the hinge u_a. The O_a point that defines the origin of these coordinate systems is selected on the pivot axis. For adjacent bodies, the vertex index of the graph s_i is $i = i^-(a)$ or $i^+(a)$. For moving coordinate systems $O_a x_{1i} x_{2i} x_{3i}$, the direction of the $O_a x_{3i}$ axis coincides with the vector $\mathbf{e}_n$. For adjacent bodies, the relative rotation of the body with the graph vertex index $i^+(a)$ relative to the body with the graph vertex index $i^-(a)$ is determined by the angle φ_a, and the relative angular velocity by the vector $\omega_a = \dot{\varphi}_a \mathbf{e}_a$. The relative rotation matrix is:

$$A_a = \begin{pmatrix} \cos\varphi_a & \sin\varphi_a & 0 \\ -\sin\varphi_a & \cos\varphi_a & 0 \\ 0 & 0 & 1 \end{pmatrix} = A(\varphi_a),\; a = \overline{1,n}.$$

Consider a vertex of the graph s_i for which there are arcs u_a, u_b and the equalities $i = i^-(a) = i^+(b)$ hold. It is required to find the direction cosine matrix A_{ab}, connecting the moving coordinate systems rigidly connected with bodies with the indices of the graph vertices $i^-(a)$ and $i^+(b)$, respectively.

LEMMA 1. The matrix A_{ab} has the form

$$A_{ab} = \begin{pmatrix} \frac{1+e_{3ab}-e_{1ab}^2}{1+e_{3ab}}\cos\alpha_{ab} + \frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\sin\alpha_{ab} & -\frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\cos\alpha_{ab} - \frac{1+e_{3ab}-e_{2ab}^2}{1+e_{3ab}}\sin\alpha_{ab} & -e_{1ab}\cos\alpha_{ab} + e_{2ab}\sin\alpha_{ab} \\ -\frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\cos\alpha_{ab} + \frac{1+e_{3ab}-e_{1ab}^2}{1+e_{3ab}}\sin\alpha_{ab} & \frac{1+e_{3ab}-e_{2ab}^2}{1+e_{3ab}}\cos\alpha_{ab} - \frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\sin\alpha_{ab} & -e_{2ab}\cos\alpha_{ab} - e_{1ab}\sin\alpha_{ab} \\ e_{1ab} & e_{2ab} & e_{3ab} \end{pmatrix}.$$

Here, in the coordinate system $O_a x_{1a} x_{2a} x_{2a}$ the vector $\mathbf{e}_b$ has the representation $\mathbf{e}_b = \begin{pmatrix} e_{1ab} \\ e_{2ab} \\ e_{3ab} \end{pmatrix}$, α_{ab} are angles determined by the choice of the direction of the coordinate axes $O_b x_{1b}$, $O_b x_{2b}$.

Proof. The representation of the matrix A_{ab} in the Rodrigues-Hamilton parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3 (\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2 = 1)$ has the form [3]

$$A_{ab} = \begin{pmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(-\lambda_0\lambda_3+\lambda_1\lambda_2) & 2(\lambda_1\lambda_3+\lambda_0\lambda_2) \\ 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & \lambda_0^2+\lambda_2^2-\lambda_1^2-\lambda_3^2 & 2(-\lambda_0\lambda_1+\lambda_2\lambda_3) \\ 2(-\lambda_0\lambda_2+\lambda_1\lambda_3) & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{pmatrix}.$$

Let $\mathbf{i}_{1b}$, $\mathbf{i}_{2b}$, $\mathbf{i}_{3b}$ be the unit vectors of the coordinate system $O_b x_{1b} x_{2b} x_{3b}$. We have $A_{ab}^T = (\mathbf{i}_{1b}, \mathbf{i}_{2b}, \mathbf{i}_{3b})$, $\mathbf{i}_{3b} = \mathbf{e}_b$. Rodrigues-Hamilton parameters are found from the system of equations

$$\begin{aligned} 2(-\lambda_0\,\lambda_2 + \lambda_1\,\lambda_3) &= e_{1ab}, \\ 2(\lambda_0\,\lambda_1 + \lambda_2\,\lambda_3) &= e_{2ab}, \\ \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 &= e_3. \end{aligned}$$

From the first two equations of this system, we obtain

$$\lambda_1 = \frac{e_{1ab}\lambda_3 + e_{2ab}\lambda_0}{2(\lambda_0^2 + \lambda_3^2)}, \quad \lambda_2 = \frac{e_{2ab}\lambda_0 - e_{1ab}\lambda_3}{2(\lambda_0^2 + \lambda_3^2)}.$$

For the variable $x = \lambda_0^2 + \lambda_3^2$ from the third equation of the system we have

$$x^2 - e_{3ab}x - \frac{1}{4}(e_{1ab}^2 + e_{2ab}^2) = 0.$$

Find the positive root of the quadratic equation $x = \frac{1}{2}(1 + e_{3ab})$. We set $\lambda_0 = \sqrt{\frac{1+e_{3ab}}{2}}\cos\frac{\alpha_{ab}}{2}$, $\lambda_3 = \sqrt{\frac{1+e_{3ab}}{2}}\sin\frac{\alpha_{ab}}{2}$ and find the rest of the Rodrigues-Hamilton parameters

$$\lambda_1 = \frac{e_{1ab}\sin\frac{\alpha_{ab}}{2} + e_{2ab}\cos\frac{\alpha_{ab}}{2}}{\sqrt{2(1+e_{3ab})}},$$
$$\lambda_2 = \frac{e_{2ab}\sin\frac{\alpha_{ab}}{2} - e_{1ab}\cos\frac{\alpha_{ab}}{2}}{\sqrt{2(1+e_{3ab})}}.$$

The dependence of the solution on the parameter α_{ab} is associated with the arbitrariness in the choice of the direction of the axes of the coordinate system $O_b x_{1b} x_{2b} x_{3b}$. It is easy to verify that the normalization condition for the Rodrigues-Hamilton parameters is satisfied. For unit unit vectors of the moving coordinate system $O_b x_{1b} x_{2b} x_{3b}$, the following formulas hold:

$$\mathbf{i}_{1b} = \begin{pmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \\ 2(-\lambda_0\lambda_3 + \lambda_1\lambda_2) \\ 2(\lambda_0\lambda_2 + \lambda_1\lambda_3) \end{pmatrix}, \quad \mathbf{e}_{2b} = \begin{pmatrix} 2(\lambda_0\lambda_3 + \lambda_1\lambda_2) \\ \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \\ 2(-\lambda_0\lambda_1 + \lambda_2\lambda_3) \end{pmatrix}.$$

After transformations we get

$$\mathbf{i}_{1b} = \begin{pmatrix} \frac{1+e_{3ab}-e_{1ab}^2}{1+e_{3ab}}\cos\alpha_{ab} + \frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\sin\alpha_{ab} \\ -\frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\cos\alpha_{ab} - \frac{1+e_{3ab}-e_{2ab}^2}{1+e_{3ab}}\sin\alpha_{ab} \\ -e_{1ab}\cos\alpha_{ab} + e_{2ab}\sin\alpha_{ab} \end{pmatrix},$$

$$\mathbf{i}_{2b} = \begin{pmatrix} -\frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\cos\alpha_{ab} + \frac{1+e_{3ab}-e_{1ab}^2}{1+e_{3ab}}\sin\alpha_{ab} \\ \frac{1+e_{3ab}-e_{2ab}^2}{1+e_{3ab}}\cos\alpha_{ab} - \frac{e_{1ab}e_{2ab}}{1+e_{3ab}}\sin\alpha_{ab} \\ -e_{2ab}\cos\alpha_{ab} - e_{1ab}\sin\alpha_{ab} \end{pmatrix}.$$

Let I_a be the set of ascending indexes of arcs connecting the vertex of the graph s_0 with the vertex s_i, $i = i^+(a)$, $\varphi = (\varphi_1, ..., \varphi_n)^T$.

LEMMA 2. The absolute angular velocity of a body with a graph vertex index $i = i^+(a)$, $1 \le a \le n$, in a moving coordinate system rigidly connected with this body is determined by the formula

$$\mathbf{\Omega_i} = \sum_{\mathbf{b}\in\mathbf{I_a}} \mathbf{B_{ab}}(\varphi)\mathbf{e_b}\dot{\varphi}_\mathbf{b},$$

where $B_{aa}(\varphi) = I_3$, $B_{ab}(\varphi) = A_{a^-}(\varphi_{a^-})\cdots A_b(\varphi_b)$, $A_b(\varphi_b) = A_{bb^+}A(\varphi_b)$; b^+ is the nearest to b index $b^+ > b$, $b \in I_a$, $b \ne a$; a^- is the nearest to a index $a^- < a$.

Proof. The absolute angular velocity $\mathbf{\Omega}_i$ is equal to the sum of the relative angular velocities $\omega_b, b \in I_a$. Each angular velocity ω_b is defined in its coordinate system rigidly connected with the body with the graph vertex index $i^+(b)$. When describing them in one coordinate system rigidly connected with the body with the graph vertex index $i^+(a)$, we use the coordinate system transformations defined above. In the considered coordinate system $\omega_a = \mathbf{e}_a\dot{\varphi}_a$. Next angular velocity $\omega_{a^-} = \mathbf{e}_{a^-}\dot{\varphi_{a^-}}$ in a coordinate system rigidly connected to the body with the vertex index graph $i^+(a^-)$. The transition to the considered coordinate system is determined by the matrix $A_{a^-}(\varphi_{a^-})$. It decomposes into a rotation by an angle φ_{a^-}, determined by the matrix $A(\varphi_{a^-})$ and a change in the axis of rotation, determined by the matrix A_{a^-a}. As a result, we have $A_{a^-}(\varphi_{a^-}) = A_{a^-a}A(\varphi_{a^-})$. Using mathematical induction, we complete the proof of the lemma.

Consider a simple open kinematic chain. In this case $I_a = \{1, ..., a\}$, $B_{ab}(\varphi) = A_{a-1}(\varphi_{a-1}) \cdots A_b(\varphi_b)$, $A_b(\varphi_b) = A_{b\,b+1}A(\varphi_b)$, $1 \le b \le a-1$.

We turn to finding the linear absolute velocities in the joints. The speed of the point O_1 of the axis of the joint connecting the mechanical system with the external body is set in the inertial coordinate system by the vector-valued function of time $\mathbf{V}_1(t)$. It is required to find the velocities $\mathbf{V}_a$, $a = 2, ..., n$, at the remaining hinge points of the mechanical system.

LEMMA 3. The absolute linear velocity of the point O_a is determined by the formula

$$\mathbf{V}_a = B_{a\,1}(\varphi)\mathbf{V}_1(t) - \sum_{c \in I_{a^-}} \mathbf{C}_{a\,c}(\varphi)\dot{\varphi}_c,$$

where $\mathbf{C}_{a\,c}(\varphi) = \sum_{b \in I_{a^-} \setminus I_{c^-}} B_{a\,b}(\varphi)B_{b\,c}(\varphi)[\mathbf{e}_c, \mathbf{d}_{b\,b^+}]$, $a \ge 2$, $\mathbf{d}_{b\,b^+} = \overrightarrow{O_bO_{b^+}}$ is a constant vector in the coordinate system $O_bx_{1\,b}x_{2\,b}x_{3\,b}$.

Proof. Using the formula for the distribution of linear velocities in an absolute solid, we have

$$\mathbf{V}_{b^+} = \mathbf{V}_b - [\mathbf{\Omega}_b, \mathbf{d}_{b\,b^+}].$$

Using mathematical induction, we find

$$\mathbf{V}_a = \mathbf{V}_1(t) - \sum_{b \in I_{a^-}} [\mathbf{\Omega}_b, \mathbf{d}_{b\,b^+}].$$

In the moving coordinate system $O_ax_{1\,a}x_{2\,a}x_{3\,a}$ the last formula is converted to the form

$$\mathbf{V}_a = B_{a\,1}(\varphi)\mathbf{V}_1(t) - \sum_{b \in I_{a^-}} B_{a\,b}(\varphi)[\mathbf{\Omega}_b, \mathbf{d}_{b\,b^+}].$$

Using the representation of angular velocities, we obtain

$$\mathbf{V}_a = B_{a\,1}(\varphi)\mathbf{V}_1(t) - \sum_{b \in I_{a^-}} B_{a\,b}(\varphi)[\sum_{C \in I_b} B_{b\,c}(\varphi)\mathbf{e}_c, \mathbf{d}_{b\,b^+}]\dot{\varphi}_c = B_{a\,1}(\varphi)\mathbf{V}_1(t) - \sum_{c \in I_{a^-}} \sum_{b \in I_{a^-} \setminus I_{c^-}} B_{a\,b}(\varphi)B_{b\,c}(\varphi)[\mathbf{e}_c, \mathbf{d}_{b\,b^+}]\dot{\varphi}_c.$$

For a simple open kinematic chain, we have

$$\mathbf{C}_{a\,c}(\varphi) = \sum_{b=c}^{a-1} B_{n\,b}(\varphi)B_{b\,c}(\varphi))[\mathbf{e}_c, \mathbf{d}_{b\,b+1}], 1 \le c \le a-1, a \ge 2.$$

KINETIC ENERGY OF A MANIPULATING ROBOT

To compose the Lagrange equations of the second kind, it is necessary to calculate the kinetic energy of the mechanical system T. The formula takes place

$$T = \sum_{i=1}^{n} T_i,$$

where T_i is the kinetic energy of the i -th body, to calculate which we use the formula [3]

$$T_i = \frac{1}{2}m_iV_a^2 + m_i([\mathbf{V}_a, \mathbf{\Omega}_i], \overrightarrow{O_aC_i}) + \frac{1}{2}(J_i\mathbf{\Omega}_i, \mathbf{\Omega}_i),$$

where m_i is the mass, C_i is the center of mass, J_i is the tensor of inertia at the point O_a, $i = i^+(a)$, $1 \le a \le n$. In the coordinate system $O_ax_{1\,i}x_{2\,i}x_{3\,i}$ vector $\overrightarrow{O_aC_i} = -\mathbf{d}_{i\,i}$ and tensor J_i are constant.

THEOREM 1. The kinetic energy of a mechanical system is determined by the formula

$$T = \frac{1}{2}\left(\alpha_0(t,\varphi) - 2\sum_{b=1}^{n} \alpha_b(t,\varphi)\dot{\varphi}_b + \sum_{b,c=1}^{n} \alpha_{b\,c}(\varphi)\dot{\varphi}_b\dot{\varphi}_c\right),$$

where the coefficients α_0, α_b, α_{bc}, $b, c = \overline{1, n}$ are determined by the formulas

$$\alpha_0(t,\varphi) = \sum_{a=1}^{n} m_a(B_{a\,1}^T(\varphi)B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{V}_1(t)),$$

$$\alpha_b(t,\varphi) = \sum_{a=1}^{n} m_a \left(S_{b\,a}^{-1}([B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}\mathbf{e}_b], \mathbf{d}_{b\,b}) + S_{b\,a^-}^{-1}(B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{C}_{a\,b}(\varphi))\right),$$

$$\alpha_{b\,c}(\varphi) = \sum_{a=2}^{n} m_a S_{c\,a^-}^{-1}\left(S_{b\,a^-}^{-1}(\mathbf{C}_{a\,b}(\varphi), \mathbf{C}_{a\,c}(\varphi)) + 2S_{b\,a}^{-1}([\mathbf{C}_{a\,c}(\varphi), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\right) + \sum_{a=1}^{n} S_{b\,a}^{-1}S_{c\,a}^{-1}(J_a B_{a\,b}(\varphi)\mathbf{e}_b, B_{a\,c}(\varphi)\mathbf{e}_c).$$

Proof. Taking into account the representation of velocities, we find the terms in the kinetic energy T_i, $i = i^+(a) = a$. We have

$$V_a^2 = (\mathbf{V}_a, \mathbf{V}_a) = (B_{a\,1}(\varphi\mathbf{V}_1(t)), B_{a\,1}(\varphi\mathbf{V}_1(t))) - 2\left(B_{a\,1}(\varphi\mathbf{V}_1(t)), \sum_{c\in I_{a^-}} \mathbf{C}_{a\,c}(\varphi)\dot{\varphi}_c\right) + \left(\sum_{b\in I_{a^-}} \mathbf{C}_{a\,b}(\varphi)\dot{\varphi}_b, \sum_{c\in I_{a^-}} \mathbf{C}_{a\,c}(\varphi)\dot{\varphi}_c\right)$$

$$= \left(B_{a\,1}^T(\varphi)B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{V}_1(t)\right) - 2\sum_{c\in I_{a^-}}(B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{C}_{a\,c}(\varphi))\,\dot{\varphi}_c + \sum_{b,c\in I_{a^-}}(\mathbf{C}_{a\,b}(\varphi), \mathbf{C}_{a\,c}(\varphi))\,\dot{\varphi}_b\dot{\varphi}_c, a \geq 2.$$

Using the properties of the incidence matrix S, we find

$$V_a^2 = \left(B_{a\,1}^T(\varphi)B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{V}_1(t)\right) - 2\sum_{c=1}^{n} S_{c\,a^-}^{-1}(B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{C}_{a\,c}(\varphi))\,\dot{\varphi}_c + \sum_{b,c=1}^{n} S_{b\,a^-}^{-1}S_{c\,a^-}^{-1}(\mathbf{C}_{a\,b}(\varphi), \mathbf{C}_{a\,c}(\varphi))\,\dot{\varphi}_b\dot{\varphi}_c, a \geq 2.$$

We have

$$[\mathbf{V}_1(t), \mathbf{\Omega}_i] = [\mathbf{V}_1(t), \mathbf{e}_1]\dot{\varphi}_1$$

and

$$[\mathbf{V}_a(t), \mathbf{\Omega}_i] = [B_{a\,1}(\varphi)\mathbf{V}_1(t), \sum_{b\in I_a} B_{a\,b}(\varphi)\mathbf{e}_b]\dot{\varphi}_b - \sum_{b\in I_a}[B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}(\varphi)\mathbf{e}_b]\dot{\varphi}_b - [\sum_{c\in I_{a^-}} \mathbf{C}_{a\,c}(\varphi)\dot{\varphi}_c, \sum_{b\in I_a} B_{a\,b}(\varphi)\mathbf{e}_b\dot{\varphi}_b]$$

$$= \sum_{b\in I_a}[B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}(\varphi)\mathbf{e}_b]\dot{\varphi}_b - \sum_{c\in I_{a^-}}\sum_{b\in I_a}[\mathbf{C}_{a\,c}(\varphi), B_{a\,b}(\varphi)\mathbf{e}_b]\dot{\varphi}_b\dot{\varphi}_c, a \geq 2.$$

Taking into account the properties of the incidence matrix S, we obtain

$$[\mathbf{V}_a(t), \mathbf{\Omega}_i] = \sum_{b=1}^{n} S_{b\,a}^{-1}[B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}(\varphi)\mathbf{e}_b]\dot{\varphi}_b - \sum_{b,c=1}^{n} S_{c\,a^-}^{-1}S_{b\,a}^{-1}[\mathbf{C}_{a\,c}(\varphi), B_{a\,b}(\varphi)\mathbf{e}_b]\dot{\varphi}_b\dot{\varphi}_c, a \geq 2.$$

We have $\left([\mathbf{V}_1(t), \mathbf{\Omega}_i], \overrightarrow{O_1C_1}\right) = ([\mathbf{V}_1(t), \mathbf{e}_1], \mathbf{d}_{1\,1})\dot{\varphi}_1$.

$$\left([\mathbf{V}_a(t), \mathbf{\Omega}_i], \overrightarrow{O_aC_i}\right) = -\sum_{b\in I_a}([B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\,\dot{\varphi}_b + \sum_{c\in I_{a^-}}\sum_{b\in I_a}([\mathbf{C}_{a\,a}(\varphi), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\,\dot{\varphi}_c\dot{\varphi}_b.$$

Using the properties of the incidence matrix S, we find

$$\left([\mathbf{V}_a(t), \mathbf{\Omega}_i], \overrightarrow{O_aC_i}\right) = -\sum_{b=1}^{n} S_{b\,a}^{-1}([B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\,\dot{\varphi}_b + \sum_{b,c=1}^{n} S_{b\,a}^{-1}S_{c\,a^-}^{-1}([\mathbf{C}_{a\,c}(\varphi), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\,\dot{\varphi}_c\dot{\varphi}_b.$$

We obtain $(J_i\mathbf{\Omega}_i, \mathbf{\Omega}_i) = \left(J_a \sum_{b\in I_a} B_{a\,b}(\varphi)\mathbf{e}_b\dot{\varphi}_b, \sum_{c\in Ia} B_{a\,c}(\varphi)\mathbf{e}_c\dot{\varphi}_c\right) = \sum_{b,c\in I_a}(J_a B_{a\,b}(\varphi)\mathbf{e}_b, B_{a\,c}(\varphi)\mathbf{e}_c)\,\dot{\varphi}_b\dot{\varphi}_c$, $a \geq 1$. Using the properties of the incidence matrix S, we find

$$(J_i\mathbf{\Omega}_i, \mathbf{\Omega}_i) = \sum_{b,c=1}^{n} S_{b\,a}^{-1}S_{c\,a}^{-1}(J_a B_{a\,b}(\varphi)\mathbf{e}_b, B_{a\,c}(\varphi)\mathbf{e}_c)\,\dot{\varphi}_c\dot{\varphi}_b, a \geq 1.$$

Now we turn to finding the kinetic energy of a mechanical system

$$T = \frac{1}{2}(m_1 V_1^2(t) + \sum_{a=2}^{n} m_a[\Big(B_{a\,1}^T(\varphi)B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{V}_1(t)\Big) - 2\sum_{c=1}^{n} S_{c\,a^-}^{-1}\,(B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{C}_{a\,c}(\varphi))\,\dot{\varphi}_c$$

$$+\sum_{b,c=1}^{n} S_{b\,a^-}^{-1} S_{c\,a^-}^{-1}\,(\mathbf{C}_{a\,b}(\varphi), \mathbf{C}_{a\,c}(\varphi))\,\dot{\varphi}_b\dot{\varphi}_c] - 2m_1([\mathbf{V}_1(t), \mathbf{e}_1], \mathbf{d}_{1\,1})\dot{\varphi}_1 + 2\sum_{a=2}^{n} m_a[-\sum_{b=1}^{n} S_{b\,a}^{-1}\,([B_{a\,1}(\varphi)\mathbf{V}_1(t), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\,\dot{\varphi}_b$$

$$+\sum_{b,c=1}^{n} S_{b\,a}^{-1} S_{c\,a^-}^{-1}\,([\mathbf{C}_{a\,c}(\varphi), B_{a\,b}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})\,\dot{\varphi}_c\dot{\varphi}_b] + \sum_{a=1}^{n}\sum_{b,c=1}^{n} S_{b\,a}^{-1} S_{c\,a}^{-1}\,(J_a B_{a\,b}(\varphi)\mathbf{e}_b, B_{a\,c}(\varphi)\mathbf{e}_c)\,\dot{\varphi}_c\dot{\varphi}_b)$$

After transforming the last expression, we get the following result

$$T = \frac{1}{2}(\sum_{a=1}^{n} m_a (B_{a\,1}^T(\varphi)B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{V}_1(t)) - 2\sum_{b=1}^{n}\sum_{a=1}^{n} m_a (S_{b\,a^-}^{-1}(B_{a\,1}(\varphi)\mathbf{V}_1(t), \mathbf{C}_{a\,b}(\varphi))$$

$$+\sum_{b,c=1}^{n}[\sum_{a=2}^{n} m_a S_{c\,a^-}^{-1}(S_{b\,a^-}^{-1}(\mathbf{C}_{a\,b}(\varphi), \mathbf{C}_{a\,c}(\varphi)) + 2S_{b\,a}^{-1}([\mathbf{C}_{a\,c}(\varphi), B_{a\,c}(\varphi)\mathbf{e}_b], \mathbf{d}_{b\,b})) + \sum_{a=1}^{b} S_{b\,a}^{-1} S_{c\,a}^{-1}(J_a B_{a\,c}(\varphi)\mathbf{e}_b, B_{a\,c}(\varphi)\mathbf{e}_c)]\dot{\varphi}_b\dot{\varphi}_c).$$

The considered mechanical system is holonomic with generalized coordinates $\varphi_1, ..., \varphi_n$. Control actions are modeled by the moments of forces $M_a\mathbf{e}_a$, $a = \overline{1, n}$, applied in cylindrical hinges. Therefore, the generalized forces are the values of these moments M_a, $a = \overline{1, n}$. Using the obtained representation of the kinetic energy of the mechanical system, we find the Lagrange equation of the second kind

$$\sum_{c=1}^{n} \alpha_{b\,c}(\varphi)\ddot{\varphi}_c + \sum_{c,d=1}^{n}\left(\frac{\partial\alpha_{b\,c}(\varphi)}{\partial\varphi_d} - \frac{1}{2}\frac{\partial\alpha_{d\,c}(\varphi)}{\partial\varphi b}\right)\dot{\varphi}_c\dot{\varphi}_d + \sum_{c=1}^{n}\left(\frac{\partial\alpha_c(t,\varphi)}{\partial\varphi_b} - \frac{\partial\alpha_b(t,\varphi)}{\partial\varphi_c}\right)\dot{\varphi}_c - \frac{1}{2}\frac{\partial\alpha_0(t,\varphi)}{\partial\varphi_b} - \frac{\partial\alpha_b(t,\varphi)}{\partial t} = M_b, b = \overline{1, n}.$$

CONCLUSIONS

The paper describes an explicit analytical dependence of the kinetic energy of a mechanical system on generalized coordinates. For each coordinate, the dependence is periodic with a period of 2π and the coefficients of the kinetic energy representation are determined by the formulas given in the work through the matrices $A(\varphi_b)$, $b = \overline{1, n}$. Therefore, in the Lagrange equations, the derivatives of the kinetic energy representation coefficients with respect to generalized coordinates can be calculated analytically. The proposed technique makes it possible to use packages of analytical calculations when compiling Lagrange equations of the second kind with a large number of degrees of freedom.

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