Wave Propagation Problem In Three-Dimensional Inhomogeneous Isotropic Layer

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Abstract. The three-dimensional problem of oscillating load moving with constant speed along the upper boundary of a two-layer isotropic medium is studied. The solution is obtained as a double integral Fourier transform. Dispersion curves analysis of the problem is carried out.

The two-dimensional elastic diffusion problem for isotropic one-component layer was solved in [1]. The solution was constructed using Fourier series, Laplace time transforms and Fourier transforms in spatial coordinate. The originals of the Laplace transform were found analytically, and quadrature formulas were used to reverse the Fourier transform. Analytical expressions were obtained for the phase velocities long-wave asymptotics of Lamb waves propagating in an isotropic layer were obtained in [2]. The problem of antiplane steady-state oscillations of elastic isotropic layer (band) under the action of an oscillating distributed load was studied in [3]. It is found that surface effects have a significant impact when the layer thickness is reduced to nanoscale dimensions. The problem of energy flow propagation in isotropic layer with a thin plate fixed on its surface was considered in [4]. The uniformly distributed load oscillates on the plate surface. The numerical analysis of the effect of the density and stiffness ratio of the plate and layer on the propagation of energy flow in the layer was performed.

In this paper, we consider the three-dimensional problem of wave propagation in a two-layer isotropic medium. The area occupied by the isotropic composite layer has the form: $\Psi=\Psi_1\cup\Psi_2$, where

$$\Psi_1=\{-\infty<x_1,x_2<+\infty,0\le x_3\le H\},\quad \Psi_2=\{-\infty<x_1,x_2<+\infty,H\le x_3\le h\}$$

The problem is given by relations:

$$(\lambda_1+\mu_1)(div\underline{u}),_k+\mu_1\Delta u_k=\rho_1\ddot{u}_k,\ k=1,2,3,$$
$$(\lambda_2+\mu_2)(div\underline{v}),_k+\mu_2\Delta v_k=\rho_2\ddot{v}_k,\ k=1,2,3, \quad (1)$$

where λ_1, μ_1, λ_2, μ_2 are Lame coefficients, ρ_1, ρ_2, are material densities of the lower and upper layers respectively.

With the following boundary conditions:

$$u_3\big|_{x_3=0}=\sigma_{31}\big|_{x_3=0}=\sigma_{32}\big|_{x_3=0}=0,\ \bar{u}\big|_{x_3=H}=\bar{v}\big|_{x_3=H},\ \sigma_{3k}\big|_{x_3=H}=t_{3k}\big|_{x_3=H},\ k=1, \quad (2)$$

$$t_{31}\big|_{x_3=h}=t_{32}\big|_{x_3=h}=0,\ t_{33}\big|_{x_3=h}=\begin{cases} f(x_1-wt,x_2)e^{i\Omega t},(x_1,x_2)\in S\\ 0,(x_1,x_2)\notin S\end{cases}$$

where S is some region with a piecewise smooth boundary.

Conditions at infinity have following form:

$$\underline{u}(x_1,x_2,x_3,t)\to 0,\ \underline{v}(x_1,x_2,x_3,t)\to 0,\ R=\sqrt{x_1^{\ 2}+x_2^{\ 2}}\to\infty\ , \quad (3)$$

The steady-state oscillation mode is considered, the solution of the system (1) is sought in the form of

$$\underline{u}(x_1,x_2,x_3,t)=\underline{U}(x,y,z)e^{i\tilde{\Omega}t}\ ,\ \underline{v}(x_1,x_2,x_3,t)=\underline{V}(x,y,z)e^{i\tilde{\Omega}t}$$

in dimensionless variables:

$$x_1=\frac{x}{H},\ x_2=\frac{y}{H},\ x_3=\frac{z}{H},\ \underline{U}^H=\frac{\underline{U}}{H},\ V^H=\frac{V}{H},\quad \Omega=\frac{\tilde{\Omega}H}{c_{11}},\quad \xi=\frac{h-H}{H},$$

$$c_{1i}^2=\frac{\lambda_i+2\mu_i}{\rho_i},\ c_{2i}^2=\frac{\mu_i}{\rho_i},\ i=1,2,\ c_{ij}^{\ H}=\frac{c_{ij}}{c_{11}},\ i,j=1,2,\quad \mu=\frac{\mu_2}{\mu_1},\rho=\frac{\rho_2}{\rho_1}.$$

To satisfy the condition (3) the limit absorption principle [5] is applied, which actually leads to the replacement of Ω by $\Omega_\varepsilon = \Omega - i\varepsilon,\ 0 < \varepsilon << 1$.

We apply to the system of differential equations (1) the double Fourier transform by coordinates x_1, x_2. We seek the solution for the lower layer in such a way that the conditions on the lower boundary at x_3=0 are fulfilled automatically. For this purpose it is enough to put the following coefficients equal to zero: $C^s = D_2^c = D_1^c = 0$. To determine the remaining unknown coefficients substitute $\tilde{u}_k, \tilde{v}_k$ in the boundary conditions (2), modified with Fourier transform and find a system of equations to determine the unknown coefficients. Solving the obtained system and substituting the found coefficients into the equations, we obtain following expressions for displacements transformed by Fourier:

$$\tilde{u}_1 = \frac{1}{\Delta}\left(-i\gamma\Delta_C ch\beta_1 x_3 - \frac{1}{\beta_2}\left[\left(\beta_2{}^2 - \alpha^2\right)\Delta_{D_2} - \alpha\gamma\Delta_{D_1}\right]ch\beta_2 x_3\right),$$
$$\tilde{u}_2 = \frac{1}{\Delta}\left(-i\alpha\Delta_C ch\beta_1 x_3 + \frac{i}{\beta_2}\left[\left(\beta_2{}^2 - \gamma^2\right)\Delta_{D_1} - \alpha\gamma\Delta_{D_2}\right]ch\beta_2 x_3\right), \tag{4}$$
$$\tilde{u}_3 = \frac{1}{\Delta}\left(\beta_1\Delta_C sh\beta_1 x_3 - \left(i\gamma\Delta_{D_2} - i\alpha\Delta_{D_1}\right)sh\beta_2 x_3\right),$$

$$\tilde{v}_1 = \frac{1}{\Delta}\left(-i\gamma\Delta_{B^c} ch\eta_1 x_3 - i\gamma\Delta_{B^s} sh\eta_1 x_3 - \frac{1}{\eta_2}\left[\left(\eta_2{}^2 - \alpha^2\right)\Delta_{A_2^s} - \alpha\gamma\Delta_{A_1^s}\right]ch\eta_2 x_3 - \right.$$
$$\left. - \frac{1}{\eta_2}\left[\left(\eta_2{}^2 - \alpha^2\right)\Delta_{A_2^c} - \alpha\gamma\Delta_{A_1^c}\right]sh\eta_2 x_3\right)$$
$$\tilde{v}_2 = \frac{1}{\Delta}\left(-i\alpha\Delta_{B^c} ch\eta_1 x_3 - i\alpha\Delta_{B^s} sh\eta_1 x_3 + \frac{1}{\eta_2}\left[\left(\eta_2{}^2 - \gamma^2\right)\Delta_{A_1^s} - \alpha\gamma\Delta_{A_2^s}\right]ch\eta_2 x_3 + \right.$$
$$\left. + \frac{1}{\eta_2}\left[\left(\eta_2{}^2 - \gamma^2\right)\Delta_{A_1^c} - \alpha\gamma\Delta_{A_2^c}\right]sh\eta_2 x_3\right) \tag{5}$$
$$\tilde{v}_3 = \frac{1}{\Delta}\left(\eta_1\Delta_{B^c} sh\eta_1 x_3 + \eta_1\Delta_{B^s} ch\eta_1 x_3 - \left(i\gamma\Delta_{A_2^c} - i\alpha\Delta_{A_1^c}\right)ch\eta_2 x_3 - \right.$$
$$\left. - \left(i\gamma\Delta_{A_2^s} - i\alpha\Delta_{A_1^s}\right)sh\eta_2 x_3\right)$$

Where Δ is system determinant, and Δ_j are determinants corresponding to unknown coefficients, $E_1 = 2\ell^2 - \frac{\omega^2}{c_{21}{}^2}$, $E_2 = \frac{c_{22}{}^2}{c_{21}{}^2}\left(2\ell^2 - \frac{\rho\omega^2}{c_{22}{}^2}\right)$, $\beta_1{}^2 = \ell^2 - \omega^2$, $\beta_2{}^2 = \ell^2 - \frac{\omega^2}{c_{21}{}^2}$, $\ell^2 = \gamma^2 + \alpha^2$, $\eta_1{}^2 = \ell^2 - \frac{\omega^2}{c_{12}{}^2}$, $\eta_2{}^2 = \ell^2 - \frac{\omega^2}{c_{22}{}^2}$.

Obviously, the solution (4), (5) makes sense if the determinant of the system Δ is not zero. The equation

$$\Delta(\gamma, \alpha, \Omega) = 0 \tag{6}$$

is called dispersion equation. The set of points in three-dimensional space defined by the dispersion equation is called the dispersion set of the problem. Using the properties of the determinants, the expression for the dispersion equation (6) when there is no dependence on α and γ separately, but only on $\alpha^2+\gamma^2=\ell^2$, can be given as:

$$\Delta(\ell,\Omega)=\left(\eta_2^2-\ell^2\right)\begin{vmatrix} 0 & 1/\eta_2 & 0 & \eta_2 & 1 & 0 & c_1 & c_2/\beta_2 & \beta_2 c_2 \\ 0 & \Delta_{22} & 0 & 0 & 0 & 0 & 0 & \Delta_{28} & 0 \\ 1 & 0 & \ell^2 & 0 & 0 & \eta_1 & \beta_1 s_1 & s_2 & \ell^2 s_2 \\ 0 & 0 & \Delta_{51} & 0 & 0 & 0 & \Delta_{47} & \Delta_{48} & \Delta_{49} \\ \Delta_{51} & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{58} & 0 \\ 0 & 0 & 0 & 2\eta_2\Delta_{51} & \Delta_{65} & 0 & \Delta_{67} & \Delta_{68} & \Delta_{69} \\ 0 & 0 & \Delta_{73} & \Delta_{74} & \Delta_{75} & 2\eta_1 c_{1\xi} & 0 & 0 & 0 \\ c_{2\xi} & s_{2\xi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_{91} & \Delta_{92} & \ell^2\Delta_{91} & \ell^2\Delta_{92} & E_2 c_{1\xi} & E_2 s_{1\xi} & 0 & 0 & 0 \end{vmatrix}$$

where $c_1=ch\beta_1$, $s_1=sh\beta_1$, $c_2=ch\beta_2$, $s_2=sh\beta_2$, $c_{1\xi}=ch\eta_1\xi$, $s_{1\xi}=sh\eta_1\xi$, $c_{2\xi}=ch\eta_2\xi$, $s_{2\xi}=sh\eta_2\xi$, $\Delta_{51}=\mu\left(\eta_2^2-\ell^2\right)$, $\Delta_{91}=2\mu\eta_2 s_{2\xi}$, $\Delta_{22}=\left(\eta_2^2-\ell^2\right)/\eta_2$, $\Delta_{92}=2\mu\eta_2 c_{2\xi}$, $\Delta_{73}=\left(\eta_2^2+\ell^2\right)c_{2\xi}$, $\Delta_{74}=\left(\eta_2^2+\ell^2\right)s_{2\xi}$, $\Delta_{65}=E_2-2\mu\eta_2^2$, $\Delta_{75}=2\eta_1 s_{1\xi}$, $\Delta_{47}=2\beta_1(1-\mu)s_1$, $\Delta_{67}=(E_1-2\mu\eta_2^2)c_1$, $\Delta_{28}=\left(\beta_2^2-\ell^2\right)c_2/\beta_2$, $\Delta_{48}=2(1-\mu)s_2$, $\Delta_{58}=\left(\beta_2^2-\ell^2\right)s_2$, $\Delta_{68}=(1-\mu\eta_2^2/\beta_2^2)c_2$, $\Delta_{49}=\left(\ell^2(1-2\mu)+\beta_2^2\right)s_2$, $\Delta_{69}=\left(\ell^2-\mu\eta_2^2\right)\beta_2 c_2$.

We present graphs of dispersion curves for different values of the problem parameters. Figure 1 shows the curves in the case where the lower layer is aluminum, the upper layer is steel, i.e. $c_{11}=1$, $c_{21}=0.525$, $c_{12}=0.831$, $c_{22}=0.452$, $\xi=0.8$, $\mu=2.062$, $\rho=2.779$. Figure 2 shows the curves in the case where the lower layer is copper, the upper layer is steel. Then the values of the problem constants are as follows: $c_{11}=1$, $c_{21}=0.596$, $c_{12}=1.54$, $c_{22}=0.838$, $\xi=0.8$, $\mu=1.661$, $\rho=0.841$.

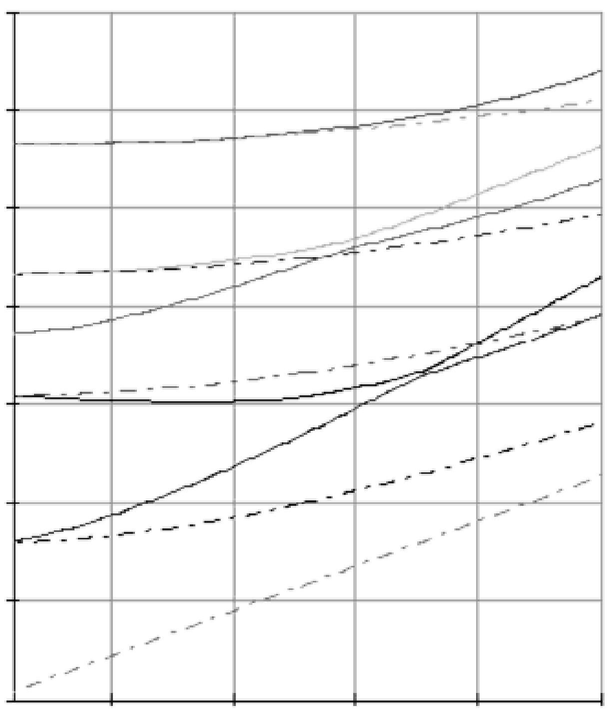

FIGURE 1.

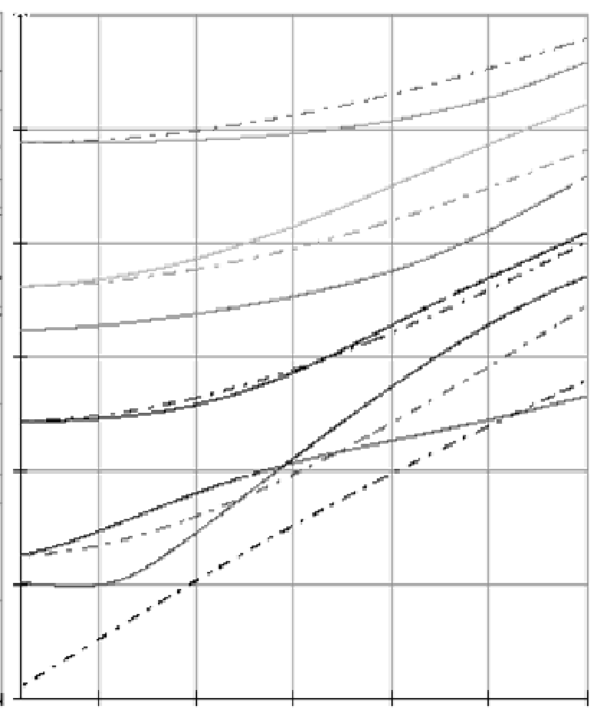

FIGURE 2.

From the figures it can be seen that choosing different materials or the ratio of their thicknesses, we get different pictures of dispersion curves. Fixing in the future the frequency of oscillation load and the speed of its movement, we can obtain a given number of propagating waves that carry energy.

Thus, the formulas for Fourier-transformed displacements are obtained, the dispersion equation of the problem is written out, the graphs of dispersion curves for various isotropic materials are plotted. This allows to carry out energy analysis of the problem for various loads acting on the surface.

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