Liesegang Operator on the Half-Line

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Abstract. We considered a Floryn problem for the "reaction-diffusion" problem on a half-line with Liesegang operator and constructed a solution to it on a certain set of time. This equation with Liesegang operator applies consideration to oscillations in colloids, which are inherit to complex colloidal systems. The existence and uniqueness theorems are been proved for this problem too.

INTRODUCTION

One often has to deal with their oscillatory behavior when considering the colloidal systems properties. Colloidal systems interact in a complex way: they are forming separate fragments (micelles), with the boundaries changing in a complex way. Oscillation is an inherent property of the colloidal systems.

On the other hand, complex colloids polymeric molecules are difficult to identify by the contemporary chemistry methods: the molecules are long and have many different isomers of different properties. Above that, colloidal systems have the property of individuality.

On the third hand, diffusion is also one of the colloidal system properties.

It follows from the properties considered above that the diffusion equation correctly defines the behavior of the colloidal system. We place in the right-hand side of the diffusion equation a term that sets the oscillations inherent to the system. We propose the so-called Liesegang operator [1, 2] as such a term. This operator has only two different values $\pm\alpha$ depending on the system development history.

We consider in this work the «reaction-diffusion» equation. The problem is considered as a problem with a moving boundary - as the Floryn problem [3] since the Liesegang operator sets not only oscillations, but also the propagating waves front motion.

The existence and uniqueness of the solution are proved for the Floryn problem on the half-line with the Liesegang operator on the right-hand side, and a solution to the problem is constructed on the time interval $t \in [0;t_0]$. It can be extended to the whole set $t > 0$.

MATHEMATICAL PROBLEM STATEMENT

Let a colloidal system be considered on a half-line $x \ge 0$, at every moment of time $t \ge 0$. We characterize the colloidal system by the concentration $u(x,t), x \ge 0, t \ge 0$. Let us introduce the Liesegang operator [1], which specifies the colloidal system oscillations.

Definition We call Liesegang operator $L[u(x,t)]$ at x a quantity such that $L\left[u(x,t)\right] = +\alpha$, if $u(x,t_0) \le u_{\min}$, and $\forall t \in (t_0;t_1)$ $u_{\min} \le u(x,t) < u_{\max}$, given that $u(x,t_1) \ge u_{\max}$, and $L\left[u(x,t)\right] = -\alpha$ if $u(x,t_0) \ge u_{\max}$, and $\forall t \in (t_0;t_1)$ $u_{\min} < u(x,t) \le u_{\max}$, given that $u(x,t_1) \le u_{\min}$.

Let us assume that $L\left[u(x,0)\right] = +\alpha$ to eliminate uncertainty at the initial moment time.

The colloid concentration dynamics problem takes the form taking into account the Liesegang operator:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) L\left[u(x,t)\right], \ x > 0, \ t > 0, \\ u(+\infty,t) = 0, t \ge 0, \ u(x,0) = f(x), \ x \ge 0, \\ u(0,t) = u_{\max}, \ t \ge 0, \ \dfrac{\partial u(0,t)}{\partial x} = 0, \ t > 0. \end{cases} \tag{1}$$

We require that the function has the following properties:

$$f(x) \in C^1[0;+\infty), \ f(0) = u_{\max}, \ f(+\infty) = 0, \ \forall x \in (0;+\infty) \ f'(x) < 0. \tag{2}$$

The function $L\left[u(x,t)\right]$ is discontinuous by definition; therefore, the classical solution to the problem, as shown in [1], does not exist. The whole set $\{x>0, \ t>0\}$ is divided by lines $u\left(x_i(t),t\right) = u_{\min}$ and $u\left(x_i(t),t\right) = u_{\max}$ according to [1], into areas such that $\forall x \in \left(x_{i+1}(t); x_i(t)\right), t \ge 0$ the function $u(x,t)$ satisfies conditions (1-2), but its derivatives $\dfrac{\partial u(x,t)}{\partial x}$ will experience a gap at the lines $x = x_i(t)$. Therefore, we define the solution as follows:

Definition The solution to the problem (1-2) is assumed to be such a function $u(x,t)$:

1. $u(x,t) \in C\left(x \in [0;+\infty), \ t \in [0;+\infty)\right)$,
2. $u(x,t) \in C^{(2,1)}\left(x \in \bigcup_{i=2}^{\infty} \left(x_{i+1}(t); x_i(t)\right) \bigcup \left(x_1(t);+\infty\right), \ t \in [0;+\infty) \right)$,
3. All the initial and boundary conditions (1-2) are satisfied,
4. Condition is satisfied: $\dfrac{\partial u\left(x_i(t)+0,t\right)}{\partial x} = 0$ on dividing lines $x = x_i(t)$ [1].

BUILDING A FORMAL SOLUTION

We restrict ourselves to building a solution on the set $x \in \left(x_1(t);+\infty\right), \ t > 0$, since, the solution construction for all individual sections is not difficult with a known first boundary law of motion. Thus, we obtain the Floryn problem [3] for $x \in \left[x_1(t);+\infty\right)$:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \alpha u(x,t), \ x > 0, \ t > 0, \\ u\left(x_1(t),t\right) = u_{\max}, t \ge 0, \dfrac{\partial u\left(x_1(t),t\right)}{\partial x} = 0, \ t \ge 0, \\ u(+\infty,t) = 0, \ t \ge 0, \ u(x,0) = f(x), \ x > 0, \ x_1(0) = 0, \ x_1(t) \in C^1(t>0). \end{cases} \tag{3}$$

We consider not a problem with a moving boundary, assuming that $x_1(t) \in C^1(t>0)$ for the convenience of the further solution:

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \alpha u(x,t) + x_1'(t)\frac{\partial u(x,t)}{\partial x}, \ x>0, \ t>0, \\
u(0,t) = u_{\max}, t \ge 0, \ \frac{\partial u(0,t)}{\partial x} = 0, \ t \ge 0, \\
u(+\infty,t) = 0, \ t \ge 0, \ u(x,0) = f(x), \ x > 0, \\
x_1(0) = 0, \ x_1(t) \in C^1(t>0).
\end{cases} \tag{4}
$$

Let us seek the solution to problem (4) in the formal series form:

$$
\begin{cases}
u(x,t) = u_0(x,t) + \sum_{j=1}^{\infty} u_j(x,t), \\
x_1(t) = v_0 t + \sum_{j=1}^{\infty}\left(\int_0^t v_j(x,\tau)\,d\tau\right), \ v_0 = const.
\end{cases} \tag{5}
$$

We obtain the following set of problems substituting expressions (5) into equations (4):

$$
\begin{cases}
\frac{\partial u_0(x,t)}{\partial t} = \frac{\partial^2 u_0(x,t)}{\partial x^2} + \alpha u_0(x,t) + v_0\frac{\partial u_0(x,t)}{\partial x}, \ x>0, \ t>0, \\
u_0(0,t) = u_{\max}, t \ge 0, \ \frac{\partial u_0(0,t)}{\partial x} = 0, \ t \ge 0, \quad u_0(+\infty,t) = 0, \ t \ge 0, \ u_0(x,0) = f_0(x), \ x > 0,
\end{cases} \tag{6}
$$

where $f_0(x) = Ae^{k_1 x} + Be^{k_2 x}$, $k_1 = -\frac{v_0}{2} + \frac{1}{2}\sqrt{v_0^2 + 4p - 4\alpha}$, $k_2 = -\frac{v_0}{2} - \frac{1}{2}\sqrt{v_0^2 + 4p - 4\alpha}$, $A = -\frac{k_2 u_{\max}}{k_1 - k_2}$, $B = \frac{k_1 u_{\max}}{k_1 - k_2}$, $\lambda = \sqrt{v_0^2 + 4p - 4\alpha}$.

For $u_1(x,t)$:

$$
\begin{cases}
\frac{\partial u_1(x,t)}{\partial t} = \frac{\partial^2 u_1(x,t)}{\partial x^2} + \alpha u_1(x,t) + v_1(t)\frac{\partial u_0(x,t)}{\partial x}, \ x>0, \ t>0, \\
u_1(0,t) = u_{\max}, t \ge 0, \ \frac{\partial u_1(0,t)}{\partial x} = 0, \ t \ge 0, \\
u_1(+\infty,t) = 0, \ t \ge 0, \ u_1(x,0) = f(x) - f_0(x), \ x > 0,
\end{cases} \tag{7}
$$

Note that in problem (7) it is necessary to choose $v_1(t)$ so that the condition of the derivative equality to zero $\frac{\partial u_1(0,t)}{\partial x} = 0 \ \ \forall t > 0$.

The following equations system is true for subsequent $u_j(x,t)$ and $v_j(t)$:

$$
\begin{cases}
\frac{\partial u_j(x,t)}{\partial t} = \frac{\partial^2 u_j(x,t)}{\partial x^2} + \alpha u_j(x,t) + v_0\frac{\partial u_j(x,t)}{\partial x} + v_j(t)\frac{\partial u_0(x,t)}{\partial x} + J_j(x,t), \ x>0, \ t>0, \\
u_j(0,t) = 0, t \ge 0, \ \frac{\partial u_j(0,t)}{\partial x} = 0, \ t \ge 0, \\
u_j(+\infty,t) = 0, \ t \ge 0, \ u_j(x,0) = 0, \ x > 0,
\end{cases} \tag{8}
$$

where the notation is used

$$J_j(x,t)=\sum_{i=1}^{j-1} v_i(t)\frac{\partial u_{j-i}(x,t)}{\partial x}. \quad (9)$$

Note that the solution to the problem (6) is found quite easily for a given function $f_0(x)$.

Lemma 1 The solution to problem (6) exists, is unique, infinitely differentiable and can be represented as:

$$u_0(x,t)=\frac{u_0 e^{pt}}{k_2-k_1}\left(e^{k_2 x}-e^{k_1 x}\right), \quad (10)$$

where $p\in\left(\alpha-v_0^2/4;\alpha\right)$.

Evidence: The fact that (10) is a solution to (6) can be verified by direct calculation.

The solution to (6) existence follows immediately from (10) and its verification. The form (10) implies its unlimited differentiability and the existence of the Laplace transform (10) [4].

Let us prove uniqueness to the solution (10). Let us suppose the opposite: let there be different solutions $u_0(x,t)$ and $v_0(x,t)$ to problem (6). Let's construct their difference $z(x,t)=u_0(x,t)-v_0(x,t)$ and transform it according to Laplace [4]: $\hat{z}(x,s)=\int_0^{+\infty} z(x,t)e^{-st}dt$. Then we get an ordinary differential equation for the function $\hat{z}(x,s)$:

$$\begin{cases} s\hat{z}(x,s)=\dfrac{d^2\hat{z}(x,s)}{dx^2}+\alpha\hat{z}(x,s)+v_0\dfrac{d\hat{z}(x,s)}{dx}, \ x>0, \ \forall s, \\ \hat{z}(0,s)=0, \ \dfrac{dz(0,s)}{dx}=0, \ \hat{z}(+\infty,s)=0, \ \forall s. \end{cases} \quad (11)$$

Problem (11) is a homogeneous Cauchy problem whose solution is zero. Consequently, we obtain that solution (11) is homogeneous, and the functions $u_0(x,t)$ and $v_0(x,t)$ match due to the one-to-one correspondence between the function image and its inverse image.

Lemma 2. The solution to problem (7) exists, is unique and can be written using the formula:

$$u_1(x,t)=\frac{u_0(\alpha-p)e^{\left(p_1 t-\frac{v_0 x}{2}\right)}}{\lambda}\int_0^t\frac{v_1(\tau)e^{\left(p-\alpha+\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{+\infty} sh(\lambda s)\left(e^{-\frac{(x-s)^2}{4(t-\tau)}}-e^{-\frac{(x+s)^2}{4(t-\tau)}}\right)ds+$$

$$+e^{\left(p_1 t-\frac{v_0 x}{2}\right)}\int_0^{+\infty}\frac{\left(f_0(s)-f(s)\right)}{\sqrt{4\pi t}}e^{\frac{v_0 s}{2}}\left(e^{-\frac{(x-s)^2}{4t}}-e^{-\frac{(x+s)^2}{4t}}\right)ds,$$

$$v_1(t)=\frac{1}{2u_0(\alpha-p)}\frac{d}{dt}\left\{\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t^3}}\int_0^{+\infty} s\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds\right\}.$$

Proof: We substitute in (7) the solutions found in Lemma 1 for $u_0(x,t)$. As a result, the basic equation (7) takes the form:

$$\frac{\partial u_1(x,t)}{\partial t}=\frac{\partial^2 u_1(x,t)}{\partial x^2}+\alpha u_1(x,t)+v_1(t)\frac{u_0 k_1 k_2}{k_2-k_1}\left(e^{k_2 x}-e^{k_1 x}\right)e^{pt}. \quad (12)$$

We look for a solution to (12) in the form $u(x,t)=e^{p_1 t-\frac{v_0}{2}x}w(x,t)$, where $w(x,t)$ is a new unknown function. We get by choosing $p_1=\alpha-v_0^2/4$ after some transformations

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2 w(x,t)}{\partial x^2}+v_1(t)\frac{u_0 k_1 k_2}{k_2-k_1}\left(e^{\lambda x}-e^{-\lambda x}\right)e^{\left(p+\frac{v_0^2}{4}-\alpha\right)t}. \quad (13)$$

Solution to problem (13) with boundary conditions $w(x,0)=e^{\frac{v_0}{2}x}\left(f_0(x)-f(x)\right)$, $\forall x>0$, $w(0,t)=0,\ t>0,$ $w(+\infty,t)=0,\ t>0$ is known [4] and has the form:

$$w(x,t)=\frac{u_0(\alpha-p)}{\lambda}\int_0^t\frac{v_1(\tau)e^{\left(p-\alpha+\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{+\infty}\sinh(\lambda s)\left(e^{-\frac{(x-s)^2}{4(t-\tau)}}-e^{-\frac{(x+s)^2}{4(t-\tau)}}\right)ds+$$
$$+\int_0^{+\infty}\frac{\left(f_0(s)-f(s)\right)}{\sqrt{4\pi t}}e^{\frac{v_0 s}{2}}\left(e^{-\frac{(x-s)^2}{4t}}-e^{-\frac{(x+s)^2}{4t}}\right)ds. \tag{14}$$

Let us choose $v_1(t)$ so that $\forall t>0$ condition $\frac{\partial u_1(0,t)}{\partial x}=0$ is true. We differentiate (14) by x, calculate the well-known integral, set $x\to 0+0.$ We get:

$$u_0(\alpha-p)e^{\lambda^2 t}\int_0^t v_1(\tau)d\tau=\frac{1}{2\sqrt{4\pi t^3}}\int_0^{+\infty}s\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds,$$

solving the equation,

$$v_1(t)=\frac{1}{2u_0(\alpha-p)}\frac{d}{dt}\left\{\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t^3}}\int_0^{+\infty}s\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds\right\}. \tag{16}$$

It is easy to see that for twice differentiable functions $f_0(x)$ and $f(x)$ function $v_1(t)$ has a singularity at zero: $\lim\limits_{t\to 0+0} v_1(t)\sqrt{t}=const$. However, in this case, integral (14) exists. By construction, (14) satisfies eq. (7) [4]; therefore, solution (7) exists and is determined by the lemma.

It is easy to see that the function $u_1(x,t)$ grows with growth t is not faster than the exponent, therefore, its Laplace transform exists. Repeating the arguments of Lemma 1, we come to the conclusion that the solution to (7) is unique.

Lemma 3. The solution to problem (8) exists, is unique and can be written in the form:

$$u_j(x,t)=\frac{u_0(\alpha-p)e^{\left(\alpha-\frac{v_0^2}{4}\right)t-\frac{v_0}{2}x}}{\lambda}\int_0^t\frac{v_j(\tau)e^{\left(p-\alpha+\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{+\infty}sh(\lambda s)\left(e^{-\frac{(x-s)^2}{4(t-\tau)}}-e^{-\frac{(x+s)^2}{4(t-\tau)}}\right)ds+$$
$$+e^{\left(\alpha-\frac{v_0^2}{4}\right)t-\frac{v_0}{2}x}\int_0^t\frac{e^{-\left(\alpha-\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{+\infty}J_j(s,\tau)e^{\frac{v_0 s}{2}}\left(e^{-\frac{(x-s)^2}{4(t-\tau)}}-e^{-\frac{(x+s)^2}{4(t-\tau)}}\right)ds,$$

$$v_j(t)=\frac{1}{2u_0(\alpha-p)}\frac{d}{dt}\left\{\frac{e^{-\lambda^2 t}}{\sqrt{4\pi}}\int_0^t\frac{e^{-\left(\alpha-\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{(t-\tau)^3}}\int_0^{+\infty}J_j(s,\tau)e^{\frac{v_0 s}{2}-\frac{s^2}{4(t-\tau)}}sds\right\}.$$

Proof: We seek solution to (8) in the form: $u_j(x,t)=w(x,t)e^{\left(\alpha-\frac{v_0^2}{4}\right)t-\frac{v_0}{2}x}$.

As a result, we obtain from the main equation (8) the problem for $w(x,t)$:

$$\frac{\partial w(x,t)}{\partial t}=v_j(t)\frac{u_0(p-\alpha)}{2\lambda}e^{\left(p-\frac{v_0^2}{4}+\alpha\right)t}\left(e^{\lambda x}-e^{-\lambda x}\right)+\frac{\partial^2 w(x,t)}{\partial x^2}+J_j(x,t)e^{-\left(\alpha-\frac{v_0^2}{4}\right)t+\frac{v_0}{2}x}, \tag{17}$$

whence, following [4], we obtain the solution to the boundary value problem (8) without taking into account the condition $\frac{\partial w(0,t)}{\partial x}=0$:

$$w(x,t)=\frac{u_0(\alpha-p)}{\lambda}\int_0^t \frac{v_j(\tau)e^{\left(p-\alpha+\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{+\infty} sh(\lambda s)\left(e^{-\frac{(x-s)^2}{4(t-\tau)}}-e^{-\frac{(x+s)^2}{4(t-\tau)}}\right)ds+$$
$$+\int_0^t \frac{e^{-\left(\alpha-\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{4\pi(t-\tau)}}\int_0^{+\infty} J_j(s,\tau)e^{\frac{v_0 s}{2}}\left(e^{-\frac{(x-s)^2}{4(t-\tau)}}-e^{-\frac{(x+s)^2}{4(t-\tau)}}\right)ds. \tag{18}$$

We get the integral equation substituting (18) into the condition $\frac{\partial w(0,t)}{\partial x}=0$ and finding form it $v_j(t)$,:

$$v_j(t)=\frac{1}{2u_0(\alpha-p)}\frac{d}{dt}\left\{\frac{e^{-\lambda^2 t}}{\sqrt{4\pi}}\int_0^t \frac{e^{-\left(\alpha-\frac{v_0^2}{4}\right)\tau}d\tau}{\sqrt{(t-\tau)^3}}\int_0^{+\infty} J_j(s,\tau)e^{\frac{v_0 s}{2}-\frac{s^2}{4(t-\tau)}}sds\right\}. \tag{19}$$

Substituting this expression into (18), we obtain the lemma statement.

Repeating the Lemma 2 proof course, we come to the conclusion that the constructed solution is unique.

ESTIMATES

Now let's find out if any x and t the series converges $u(x,t)=\sum_{j=0}^{\infty}u_j(x,t)$.

For this, we will perform at estimates number.

Lemma 4. $\exists A_1>0,\ \exists t_0>0$, what $\forall t\in[0;t_0]$ the estimate is correct: $|v_1(t)|\le\frac{A_1}{\sqrt{t}}$.

Proof: According to (16),

$$v_1(t)=\frac{1}{2u_0(\alpha-p)}\frac{d}{dt}\left\{\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t^3}}\int_0^{+\infty} s\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds\right\}.$$

Differentiating

$$v_1(t)=-\frac{\lambda^2 e^{-\lambda^2 t}}{2u_0(\alpha-p)\sqrt{4\pi t^3}}\int_0^{+\infty} s\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds-\frac{3e^{-\lambda^2 t}}{4u_0(\alpha-p)\sqrt{4\pi t^5}}\int_0^{+\infty} s\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds+$$
$$+\frac{e^{-\lambda^2 t}}{8u_0(\alpha-p)\sqrt{4\pi t^7}}\int_0^{+\infty} s^3\left(f_0(s)-f(s)\right)e^{\frac{v_0 s}{2}-\frac{s^2}{4t}}ds.$$

We restrict ourselves to considering the second term (for the first and third terms the proof is carried out in a similar way), in which, for simplicity, we omit the exponent and fraction.

Note that $f_0(0)=f(0)$, $f_0'(0)=f'(0)=0$, moreover, there is also the second derivative of both functions at zero, and both functions are bounded. Hence, $\forall x\ge 0\ \ \exists A_2>0\ \ \left|f_0(x)-f(x)\right|\le A_2 x$. We estimate the integral from above using this relation:

$$|I_2|\le\frac{4A_2}{\sqrt{\pi t}}\int_0^{+\infty}\left(\frac{s}{2\sqrt{\tau}}\right)^2 e^{-\left(\frac{s}{2\sqrt{t}}\right)^2+\frac{v_0 s}{2}}d\left(\frac{s}{2\sqrt{t}}\right)=\frac{4A_2}{\sqrt{\pi t}}\int_{-v_0\sqrt{t}/4}^{+\infty}\left(\eta+\frac{v_0}{4}\sqrt{t}\right)^2 e^{-\eta^2}d\eta.$$

Opening the brackets and performing the calculations

$$\left|I_2\right| \le \frac{3A_2 v_0}{4\sqrt{\pi t}} e^{\frac{3v_0^2 t}{16}} + \frac{2\sqrt{2}A_2}{\sqrt{t}} + \frac{v_0 A_2}{2\sqrt{2}}.$$

Quantities choice t_0 and A_2 let's get the lemma true.

Lemma 5. Let be $\forall j = 0,...,i-1$ $\exists u_j(x,t)$, such that $u_j(x,t)$ continuous and limited $\forall x \ge 0$, $\forall t \ge 0$ together with its first and second derivatives, and $\frac{\partial u_j(0,t)}{\partial x} = 0$. Let be $\forall j = 0,...,i-1$ $\exists A_j > 0$, $t_j > 0$, such that $\forall t \in [0;t_j]$ is true $\left|v_j\right| \le \frac{A_j}{\sqrt{t}}$. Then $\exists A_i > 0$, $t_i > 0$, such that $\forall t \in [0;t_i]$ is true $\left|v_i\right| \le \frac{A_i}{\sqrt{t}}$.

Proof: According to (9), $J_j(x,t) = \sum_{i=1}^{j-1} v_i(t) \frac{\partial u_{j-i}(x,t)}{\partial x}$. According to the induction hypothesis, $\left|v_j\right| \le \frac{A_j}{\sqrt{t}}$, and $\left|\frac{\partial u_j(x,t)}{\partial x}\right| \le B_j x$. Let's choose the numbers largest A_j and B_j. Hence, $\left|J_j(x,t)\right| \le \frac{A_{\max} B_{\max}}{\sqrt{t}} x$.

Substituting the resulting estimate into (19), we obtain the relation:

$$\left|v_j(t)\right| \le \frac{-A_{\max} B_{\max} \lambda^2 j e^{-\lambda^2 t}}{2\sqrt{4\pi} u_0(\alpha - p)} \int_0^t \frac{e^{-\left(\alpha - \frac{v_0^2}{4}\right)\tau} d\tau}{\sqrt{\tau}\sqrt{(t-\tau)^3}} \int_0^{+\infty} e^{\frac{v_0 s}{2} - \frac{s^2}{4(t-\tau)}} s^2 ds + \frac{3A_{\max} B_{\max} j e^{-\lambda^2 t}}{8\sqrt{\pi} u_0(\alpha - p)} \int_0^t \frac{e^{-\left(\alpha - \frac{v_0^2}{4}\right)\tau} d\tau}{\sqrt{\tau}\sqrt{(t-\tau)^5}} \int_0^{+\infty} e^{\frac{v_0 s}{2} - \frac{s^2}{4(t-\tau)}} s^3 ds +$$

$$+ \frac{A_{\max} B_{\max} j e^{-\lambda^2 t}}{8\sqrt{\pi} u_0(\alpha - p)} \int_0^t \frac{e^{-\left(\alpha - \frac{v_0^2}{4}\right)\tau} d\tau}{\sqrt{\tau}\sqrt{(t-\tau)^7}} \int_0^{+\infty} e^{\frac{v_0 s}{2} - \frac{s^2}{4(t-\tau)}} s^5 ds.$$

To prove the lemma, we choose the second integral (for the first and third proofs they are carried out in a similar way) and calculate it. The result obtained implies an estimate for small enough t:

$$\int_0^t \frac{e^{-\left(\alpha - \frac{v_0^2}{4}\right)\tau} d\tau}{\sqrt{\tau}\sqrt{(t-\tau)^5}} \int_0^{+\infty} e^{\frac{v_0 s}{2} - \frac{s^2}{4(t-\tau)}} s^2 ds \le C(v_0) \int_0^t \frac{e^{-\left(\alpha - \frac{v_0^2}{4}\right)\tau} d\tau}{\sqrt{\tau}\sqrt{t-\tau}}.$$

The integral itself is rather cumbersome. By integrating the resulting function, we obtain the desired estimate.

Lemma 6. Let be $\forall j = 0,...,i-1$ $\exists u_j(x,t)$, such that $u_j(x,t)$ is continuous and bounded $\forall x \ge 0$, $\forall t \ge 0$ together with its first and second derivatives, and $\frac{\partial u_j(0,t)}{\partial x} = 0$. Let be $\forall j = 0,...,i-1$ $\exists A_j > 0$, $t_j > 0$, such that $\forall t \in [0;t_j]$ is true $\left|v_j\right| \le \frac{A_j}{\sqrt{t}}$. Then $\exists B_j > 0$, $t_j > 0$, such that $\forall t \in [0;t_j]$ is true $\left|\frac{\partial u_j(x,t)}{\partial x}\right| \le B_j x$.

Proof: To prove this, we differentiate the expression $u_j(x,t)$ from Lemma 3 twice x, we use the estimates from the conditions and pass to the limit as $x \to 0+0$. As a result, we find out that there is a finite function $\frac{\partial^2 u_j(0,t)}{\partial x^2}$.

Theorem $\exists t_0 > 0$, such that $\forall x \ge 0$ serie $\sum_{j=0}^{\infty} u_j(x,t)$ converges uniformly $\forall t \in [0;t_0]$.

Proof: Evaluating the expression first derivative $u_j(x,t)$ from Lemma 3 to x and using the properties of $u_j(x,t)$ and of its derivatives, make sure that the uniform estimate $\left|u_j(x,t)\right| \le B_j\sqrt{t}$ is true. Row $\sum_{j=0}^{\infty} u_j(x,t)$ converges for all $B_j < 4^j$; therefore, choosing values t so that $\sqrt{t} < \frac{1}{4u_{\max}}$, we get a uniformly converging series.

CONCLUSIONS

Thus, the series we have constructed converges on the interval $t \in \left[0; \frac{1}{16u_{\max}}\right]$ at any x. For subsequent time moments, the solution can be constructed in a similar way, choosing as the initial condition the "previous" solution $f(x) = u(x,t_0)$. Thus, the solution can be constructed at any finite time in the converging series form.

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