Mathematical Modeling of Wave Parameters of the Flow of a Thin Layer of Viscous Liquid in Film Apparatuses

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Abstract. Flows of thin layers of viscous liquids (liquid films) are investigated due to wide usage of film apparatuses in many industries such as chemical, petrochemical, energy, etc. Study and calculation of liquid film wave characteristics make it possible to improve industry equipment because of better understanding how various physical and chemical factors influence on liquid film flow. For moderate Reynolds numbers, a mathematical model of the wavy flow of viscous liquid film is presented – system of Navier-Stokes and continuity equations with boundary conditions taking into account surface tension forces, surface viscosity, and thermocapillary forces. A nonlinear partial differential equation describing the state of the free surface of a liquid film is obtained. In the scope of the mathematical model, algorithms and programs are developed to calculate wave characteristics (frequency, increment, phase velocity) of liquid film.

INTRODUCTION

The study of unstable flows of liquid films began with the work by P. L. Kapitsa [1, 2]. The study of thin layers of viscous liquids (liquid films) that combine a small thickness and a large contact surface is widely implemented in various film devices of the chemical, energy, metallurgical, food, and pharmaceutical industries [3, 4, 5]. Flows of liquid films are influenced by various physical and chemical factors. The use of surfactants has a qualitative effect on flow modes [6]. Temperature and concentration gradients cause inhomogeneity of surface tension and the appearance of thermocapillary forces [7].

Relevance. The relevance of studying liquid films is due to the wide implementation of their flows in various film devices.

Novelty. The novelty is a nonlinear mathematical model of the state of the free surface of a liquid film. The model coefficients take into account the influence of thermocapillary forces and surface viscosity forces

Purpose of research. Study of the influence of various physical and chemical factors (temperature gradients, insoluble surfactants) on the wave characteristics of the liquid film.

MATHEMATICAL MODEL OF A NON-ISOTHERMAL LIQUID FILM

Let us consider the flow of a thin layer of a viscous incompressible fluid over a solid impermeable surface in a Cartesian rectangular coordinate system OXY. A mathematical model of the flow of a thick liquid film includes system of Navier-Stokes equations, the continuity equation (1), and boundary conditions (2) – (4) that take into account the influence of thermocapillary forces and surface viscosity forces, and kinematic condition (5):

$$\begin{cases} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+F_x+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}\right), \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0; \end{cases} \tag{1}$$

$$y=0: u=0, v=0; \tag{2}$$

$$y=\delta: \frac{1}{Re}\left[2\frac{\partial u}{\partial x}\frac{\partial \delta}{\partial x}-2\frac{\partial v}{\partial y}\frac{\partial \delta}{\partial x}-\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+M\frac{\partial \delta}{\partial x}+N\frac{\partial^2 u}{\partial x^2}=0; \tag{3}$$

$$P=\frac{2}{Re}\left[\frac{\partial v}{\partial y}-\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]-\overline{\sigma}\frac{\partial^2 \delta}{\partial x^2}+P_0. \tag{4}$$

$$\frac{\partial \delta}{\partial t}=v-u\frac{\partial \delta}{\partial x}. \tag{5}$$

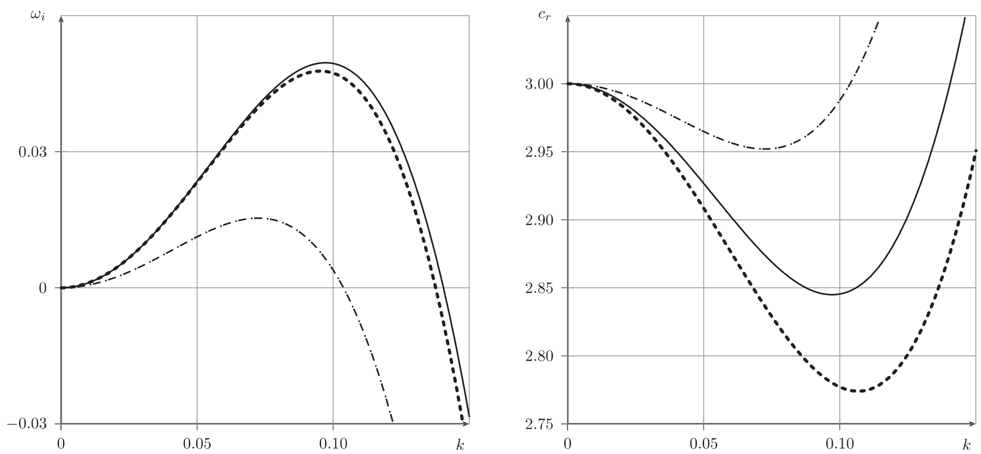

FIGURE 1. Increments ω_i and phase velocities c_r for $Re = 5$ (water): dashed curve – $M = 0$, $N = 0$; solid curve – $M = 2$, $N = 0$; dotted curve – $M = 2$, $N = 1$

In the presented model (1)–(5), dimensionless values are: u,v – projections of the film velocity on the corresponding coordinate axes; t – time; x, y – variables; δ - the thickness of the liquid film; Re – Reynolds number; Fr – Froude number; P – pressure; N – surface viscosity parameter; M – Marangoni parameter; σ – surface tension parameter.

The problem (1)–(4) is solved with the perturbation method [8]. First two approximations for the velocities u and v are found. Then, using kinematic condition (5) an equation of free surface of liquid film is obtained [7]:

$$\left(a_7\frac{\partial}{\partial x}+a_{13}\right)\frac{\partial\psi}{\partial t}+a_1\frac{\partial^4\psi}{\partial x^4}+a_4\frac{\partial^3\psi}{\partial x^3}+a_6\frac{\partial^2\psi}{\partial x^2}+a_{11}\frac{\partial\psi}{\partial x}+a_{14}\psi\frac{\partial\psi}{\partial x}+a_{16}\psi\frac{\partial^2\psi}{\partial x^2}+a_{17}\psi\frac{\partial^2\psi}{\partial x\partial t}+$$
$$+a_{21}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}+a_{22}\left(\frac{\partial\psi}{\partial x}\right)^2+a_{26}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}+a_{28}\psi\frac{\partial^3\psi}{\partial x^3}+a_{30}\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^3}+a_{34}\psi\frac{\partial^4\psi}{\partial x^4}=0 \tag{6}$$

where ψ denotes deviation of the liquid film free surface from unperturbed state, $a_1 = -\frac{\mathrm{Re}\sigma}{3}$, $a_4 = -\frac{\mathrm{Re}^2 FN}{2}$, $a_6 = -\frac{\mathrm{Re}M}{2}+\frac{3}{40}\mathrm{Re}^3F^2$, $a_7 = \frac{5}{24}\mathrm{Re}^2F$, $a_{11} = -\mathrm{Re}F$, $a_{13} = -1$, $a_{14} = -2\mathrm{Re}F$, $a_{16} = -\mathrm{Re}M+\frac{9}{20}\mathrm{Re}^3F^2$, $a_{17} = \frac{5}{6}\mathrm{Re}^2F$, $a_{21} = a_{17}$, $a_{22} = a_{16}$, $a_{26} = -\frac{1}{2}\mathrm{Re}^2FN$, $a_{28} = 3a_{26}$, $a_{30} = a_{34} = -\mathrm{Re}\sigma$.

Differential equation (6) is the mathematical model of state of non-isothermal liquid film.

COMPUTATIONAL EXPERIMENTS

The aim of our computational experiments is study of instability of water film flow for Reynolds numbers $Re \leqslant 10$. The research is performed with MATLAB software.

We consider linear part of Eqs. (6)

$$\left(a_7\frac{\partial}{\partial x}+a_{13}\right)\frac{\partial\psi}{\partial t}+a_1\frac{\partial^4\psi}{\partial x^4}+a_4\frac{\partial^3\psi}{\partial x^3}+a_6\frac{\partial^2\psi}{\partial x^2}=0. \tag{7}$$

We set $\psi = Aexp(kx-\omega t)$ and obtain dispersion relation [7] as follows

$$\omega(a_7k+i)+a_1k^4-a_4ik^3-a_6k^2=0. \tag{8}$$

In Eqs. (8), A denotes the wave amplitude, k is wave number, $\omega = \omega_r + i\omega_i$, ω_r is frequency, ω_i is increment. Phase velocity is introduced as follows $c_r = \frac{\omega_r}{k}$.

Results of computational experiments are demonstrated in Fig. 1 – 3. In Fig. 1 – 3, curves 1 correspond to free flow of water film ($M = 0, N = 0$), curves 2 show effect of temperature gradients ($M = 2, N = 0$), and curves 3 demonstrate joined effect of Marangoni parameter and insoluble surfactants ($M = 2, N = 1$).

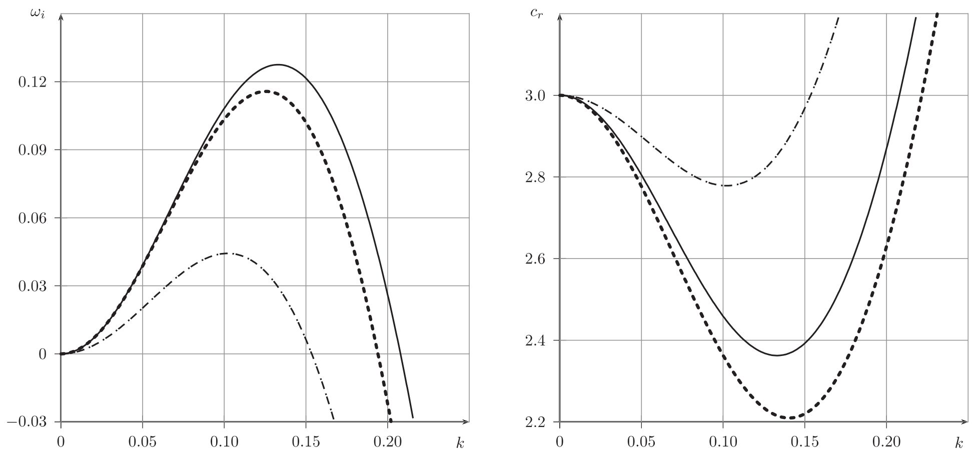

FIGURE 2. Increments ω_i and phase velocities c_r for $Re = 8$ (water): dashed curve – $M = 0$, $N = 0$; solid curve – $M = 2$, $N = 0$; dotted curve – $M = 2$, $N = 1$

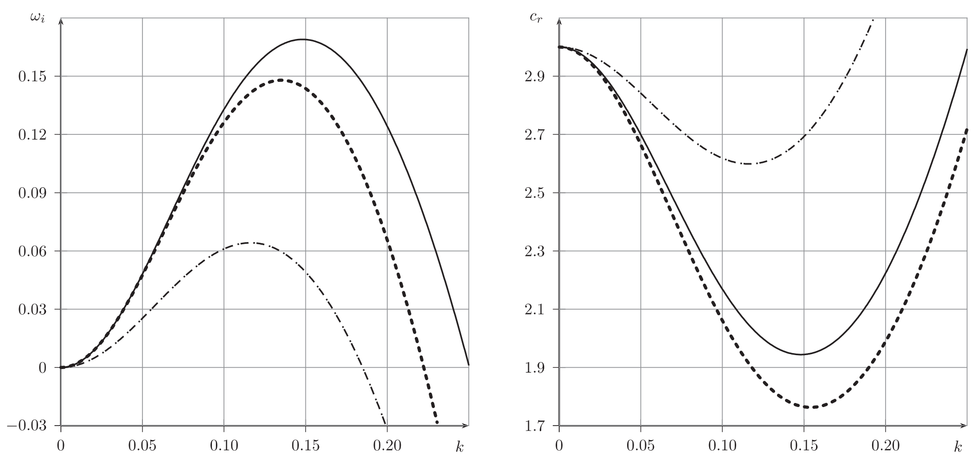

FIGURE 3. Increments ω_i and phase velocities c_r for $Re = 10$ (water): dashed curve – $M = 0$, $N = 0$; solid curve – $M = 2$, $N = 0$; dotted curve – $M = 2$, $N = 1$

Temperature and concentration gradients result in significant grow of the liquid film increments in the range of $Re \leqslant 10$. Phase velocities c_r get smaller.

Appending of insoluble surfactants leads to increments decreasing of water film. However, phase velocities get smaller. Curves 3 in Fig. 1 – 3 demonstrate wave characteristics of water film in case of joined effect of Marangoni parameter M and surface viscosity parameter N.

Wave numbers k for which increment ω_i has positive values set the instability area of the liquid film. Effect of temperature gradients leads to the instability area expansion. Surfactans aid in decreasing of the area instability. Fig. 4 shows neutral stability curves corresponding to the wave numbers for which $\omega_i = 0$. The curves split areas of stable and unstable flow of liquid film.

Increment ω_i has a maximum $k_{\omega_{i\,max}}$ calculated in Table I. The modes corresponding to $k_{\omega_{i\,max}}$ are realised with the most probability in experiments [3]. The more Marangoni parameter M the less phase velocities at $k_{\omega_{i\,max}}$. Surface viscosity parameter N results in more decreasing of phase velocities at $k_{\omega_{i\,max}}$.

We calculate wave profiles of the free surface of water film. We solve Eqs. (6) numerically with finite differences

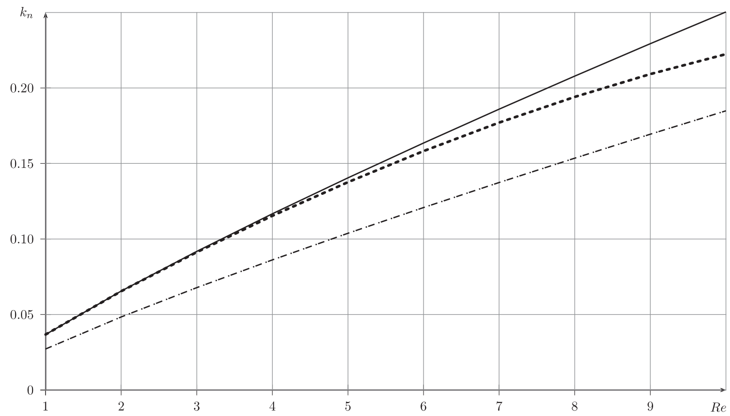

FIGURE 4. Curves of neutral stability for water film: dashed curve – $M=0$, $N=0$; solid curve – $M=2$, $N=0$; dotted curve – $M=2$, $N=1$

TABLE I. Wave number at maximal ω_i and corresponding phase velocity

Re	$M=0, N=0$		$M=2, N=0$		$M=2, N=1$	
	$k_{\omega_{i\,max}}$	$c_r(k_{\omega_{i\,max}})$	$k_{\omega_{i\,max}}$	$c_r(k_{\omega_{i\,max}})$	$k_{\omega_{i\,max}}$	$c_r(k_{\omega_{i\,max}})$
1	0.01919	2.99986	0.02598	2.99954	0.02597	2.99852
2	0.03418	2.99825	0.04626	2.99412	0.04618	2.98775
3	0.04784	2.99230	0.06467	2.97431	0.06433	2.95596
4	0.06058	2.97823	0.08165	2.92816	0.08074	2.89061
5	0.07245	2.95201	0.09714	2.84494	0.09524	2.78267
6	0.08340	2.91046	0.11096	2.71955	0.10760	2.63081
7	0.09331	2.85229	0.12292	2.55525	0.11765	2.44284
8	0.10211	2.77860	0.13300	2.36245	0.12543	2.23257
9	0.10974	2.69265	0.14133	2.15460	0.13113	2.01489
10	0.11629	2.59882	0.14810	1.94407	0.13502	1.80198

method. In Eqs. (6), the terms are approximated as follows

$$\left.\frac{\partial^4\psi}{\partial x^4}\right|_i^j \approx \frac{\psi_{i+2}^j - 4\psi_{i+1}^j + 6\psi_i^j - 4\psi_{i-1}^j + \psi_{i-2}^j}{\Delta x^4},$$

$$\left.\frac{\partial^3\psi}{\partial x^3}\right|_i^j \approx \frac{\psi_{i+2}^j - 3\psi_{i+1}^j + 3\psi_{i-1}^j - \psi_{i-2}^j}{2\Delta x^3},$$

$$\left.\frac{\partial^2\psi}{\partial x^2}\right|_i^j \approx \frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{\Delta x^2},$$

$$\left.\frac{\partial\psi}{\partial x}\right|_i^j \approx \frac{\psi_{i+1}^j - \psi_{i-1}^j}{2\Delta x},$$

$$\left.\frac{\partial\psi}{\partial t}\right|_i^j \approx \frac{\psi_i^{j+1} - \psi_i^j}{\Delta t},$$

$$\left.\frac{\partial^2\psi}{\partial x\partial t}\right|_i^j \approx \frac{1}{\Delta t}\left(\frac{\psi_{i+1}^{j+1} - \psi_i^{j+1}}{\Delta x} - \frac{\psi_{i+1}^j - \psi_i^j}{\Delta x}\right).$$

where $\psi_i^j = \psi(x_i, t_j)$, x_i denotes a node inside the computational domain, t_j denotes time step.

Our calculations show that for the Reynolds numbers Re $\leqslant 10$ liquid film achieves a periodical wavy mode of flow which is characterised by a constant amplitude A (Fig. 5, 6).

Effect of Marangoni parameter M demonstrates that wave profiles get more complex in comparing with free flow

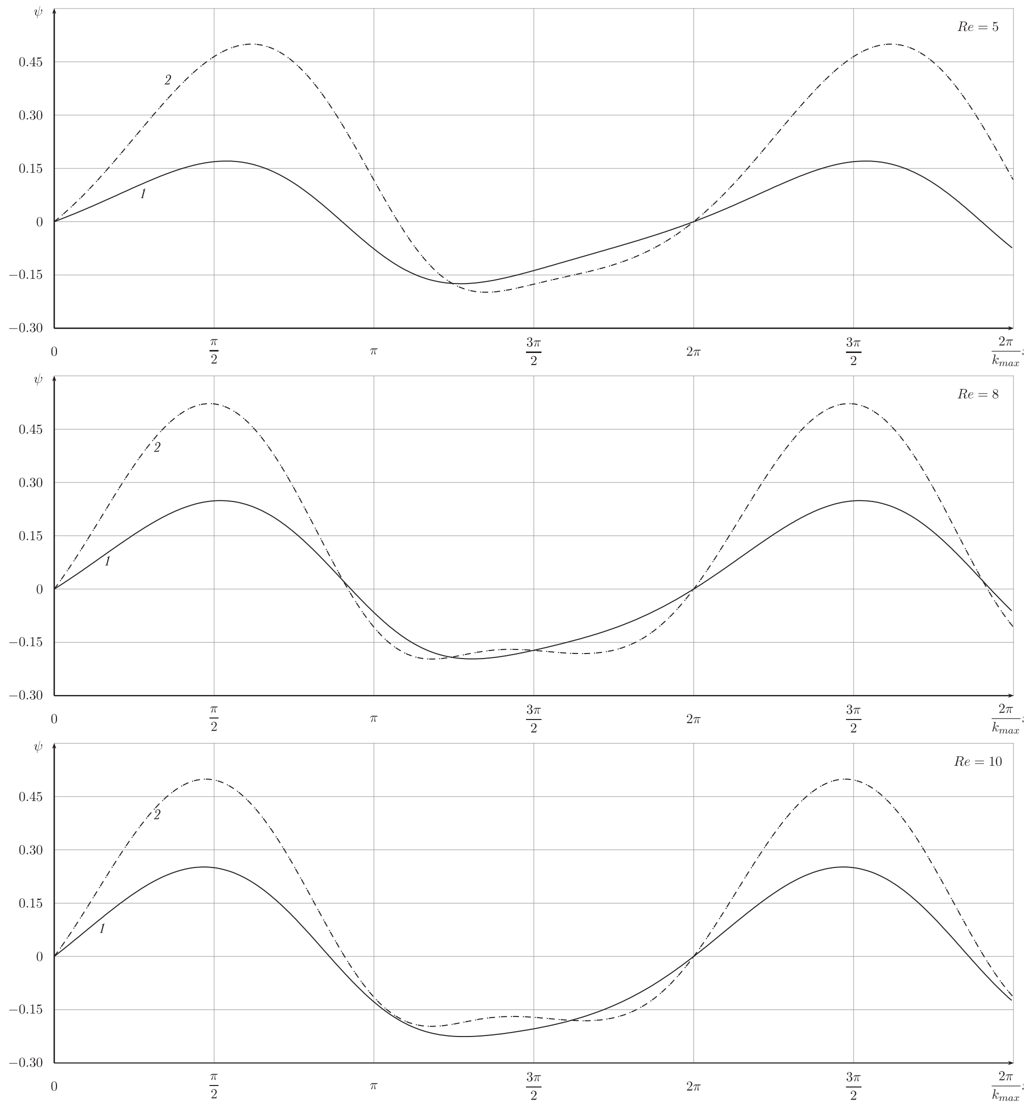

FIGURE 5. Wave shapes on free surface of water film: *1* – $M = 0$; *2* – $M = 2$

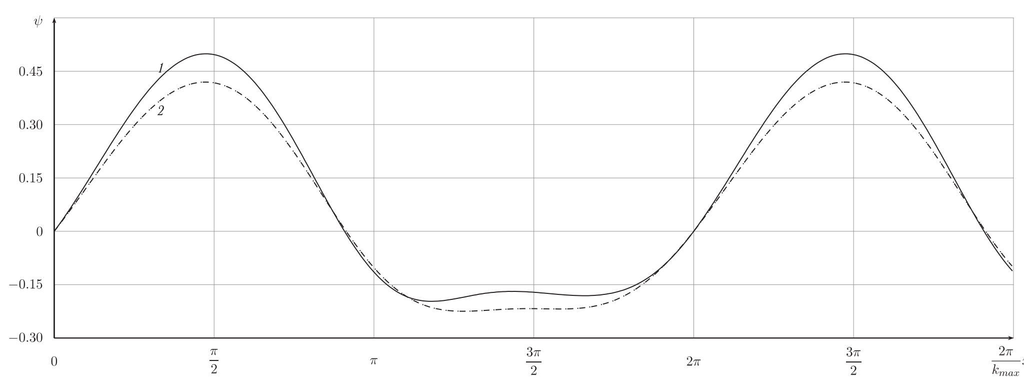

FIGURE 6. Wave profile of water film (Re = 10) when joined effect of Marangoni parameter and surfactants: *1* – $M = 2$, $N = 0$; *2* – $M = 2$, $N = 1$

of liquid film ($M = 0$, $N = 0$). The wave amplitudes increase under effect of temperature and concentration gradients (Fig. 5).

Fig. 6 demonstrates that insoluble surfactants lead to the wave amplitudes decrease, the wave profile gets smoother.

CONCLUSION

A nonlinear mathematical model of the flow of a non-isothermal liquid film for Reynolds numbers $\mathrm{Re} \leqslant 10$ is presented.

A nonlinear differential equation of the state of the free surface of a liquid film is presented. The wave parameters of the liquid film are calculated using this equation.

Using the MATLAB software, a computer simulation of the state of the free surface of the vertical water film was performed, taking into account temperature gradients and insoluble surfactants.

When liquid film flows down on heated wall, temperature gradients lead to increments increasing and instability area expansion, phase velocities get smaller. We observer wave amplitudes grow.

Adding of high viscosity surfactants causes the liquid film stabilization, results in decreasing of increments and phase velocities, restriction of area instability. Wave amplitudes become smaller.

REFERENCES

1. P. L. Kapitsa and S. Kapitsa, "Wave flow of thin layers of viscous fluid," Zh. Eksper. Teor. Fiz. **18**, 3–28 (1948).
2. P. L. Kapitsa and S. Kapitsa, "Wave flow of thin layers of viscous fluid," Zh. Eksper. Teor. Fiz. **19**, 105–120 (1949).
3. L. Kholpanov and V.Ya.Shkadov, *Hydrodynamics and heat-and-mass transfer with an interface surface (in Russian)* (Nauka, 1990) p. 341.
4. S. V. Alekseenko, V. E. Nakoryakov, and B. G. Pokusaev, *Wave flow of liquid films* (Begell House, 1994) p. 313.
5. L. A. Prokudina and G. P. Vyatkin, "Instability of a nonisotermal liquid film," Doklady Physics **43**, 652–654 (1998).
6. G. A. Filippov, G. A. Saltanov, and A. N. Kukushkin, *Hydrodynamics and heat-and-mass transfer in the presence of surfactants (in Russian)* (Begell House, 1988) p. 184.
7. L. A. Prokudina, "Nonlinear development of the marangoni instability in liquid films," J. Engng Phys. Thermophys. **89**, 921–928 (2016).
8. A. H. Nayfeh, *Perturbation methods* (Wiley-VCH, 1973) p. 448.